ANALYTICAL AND NUMERICAL SOLUTIONS OF A MIXED PROBLEM FOR THE GENERALIZED EQUATIONS OF PRANDTL

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There are, at the present time, many papers devoted to an examination and numerical solution of various types of equations of motion of a viscous fluid, these equations being in some sense much simpler than the Navier–Stokes equations (see, for example, [1, 2]). The choice of terms retained in the various modifications of the latter equations is usually dictated by the physical properties present in a specific problem. A study was made in [3] of the stability of the solutions of the generalized Prandtl equations in terms of the curl of the velocity.

We study a two-dimensional stationary flow of a viscous fluid in which a fundamental direction has been chosen for the flow along solid surfaces. As an example, we cite the case of flow in ducts. There the equations of motion must satisfy two basic requirements: They must describe the flow close to the boundary and also in the interior of the fluid, and they must display the evolutionary nature of the equations in the direction of the longitudinal coordinate. The equations can be obtained formally by retaining terms of order $(\mathrm{Re})^{-1/2}$ in the derivation of the Prandtl equations. The terms neglected have a high order of smallness close to the boundary and also in the fluid interior.

We then have (employing the usual notation)

$$\begin{gathered} u\partial u/\partial x + v\partial u/\partial y = -\partial p/\partial x + (1/\mathrm{Re})\partial^2 u/\partial y^2; \\ u\partial v/\partial x + v\partial v/\partial y = -\partial p/\partial y + (1/\mathrm{Re})\partial^2 v/\partial y^2; \\ \partial u/\partial x + \partial v/\partial y = 0. \end{gathered} \tag{1}$$

The Reynolds number Re is determined here, for example, from the characteristic dimension of the flow region and from the mean velocity. The exterior and interior asymptotic expansions of the system of equations (1) reduce, as do also the Navier–Stokes equations, to the Euler equations and to the Prandtl equations, respectively. Thus, having a solution of the system (1), we can bypass the matching process. Moreover, we can rewrite Eqs. (1) in the form of the Cauchy–Kovalevskaya equations:

$$\begin{gathered} \partial p/\partial x = (1/\mathrm{Re})\partial^2 u/\partial y^2 + u\partial v/\partial y - v\partial u/\partial y; \\ \partial v/\partial x = (1/\mathrm{Re})\partial^2 v/\partial y^2 - (v/u)\partial v/\partial y - (1/u)\partial p/\partial y; \\ \partial u/\partial x = -\partial v/\partial y; \end{gathered} \tag{2}$$

these equations make it possible to carry out the flow calculations from the initial section downstream. We point out two obvious advantages in using the system of equations (2) to make calculations in comparison with a solution of the Navier–Stokes equations: economy in machine time and the absence of any need to assign the flow parameters at the exit section, often a troublesome item. In what follows we examine the group-invariant solutions of the system (1), linearized solutions of this system, and a Cauchy problem for the system (2).

In the system of differential equations (1) the number m of unknown functions is $m=3$, and the number of independent variables is $n=2$. We seek a basis for the Lie algebra of infinitesimal operators of the group of transformations admissible by the system of equations (1). To proceed, we construct in accordance with the general theory given in [4] a second extension of an arbitrary operator of the group and then solve the system of defining equations. As a result, we obtain a basis L_4 (R turns out to be equal to 4) in the form

$$X_1 = -x\partial/\partial x - y\partial/\partial y + u\partial/\partial u + v\partial/\partial v + 2p\partial/\partial p,\ X_2 = \partial/\partial x,\ X_3 = \partial/\partial y,\ X_4 = \partial/\partial p.$$

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The general rank of the matrix $\|\xi^i_\alpha, \eta^k_\alpha\|$, i=1, 2; k=1,..., 3; α=1,..., 4, is equal to 4; it is preserved on a manifold specified by the solutions of the system of equations (1). Let r denote the rank of the corresponding Lie subalgebra. Any nonsingular invariant manifold of the group G_4 is given in the form

$$\Psi^\nu(I_1,\ldots, I_t) = 0, \ \nu = 1, 2, \ldots, \mu.$$

Then n + m − r = t; the number $\rho = t - \mu$ ($\rho = t - 3$) is called the rank of a partially invariant (of an invariant) solution, and the number $\delta = m - \mu$ is the invariance defect. A necessary condition for the existence of a partially invariant solution may, in this case, be written in the form

$$\max\{r-2, 0\} \leqslant \delta \leqslant \min\{r-1, 2\},$$

making it possible to construct a table of the possible types of partially invariant (invariant) solutions of Eqs. (1) (see [5]). With the aid of the matrices of the inner automorphisms of the algebra L_4, we determine an optimal system of one-parameter subgroups θ_1, which has, in this case, the form

$$X_1, \ \alpha X_2 + \beta X_3 + \gamma X_4.$$

Of greatest interest is the invariant solution 1, constructed on the subgroup H_1. We have the following set of invariants:

$$I_1 = ux, \ I_2 = vx, \ I_3 = px^2, \ I_4 = y/x.$$

In accordance with this, we seek a solution of the system (1) in the form

$$u = (1/x)\varphi(\xi), \ v = (1/x)\psi(\xi), \ p = p_0 + (1/x^2)\pi(\xi), \ \xi = y/x. \quad (3)$$

Substituting the expression (3) into Eqs. (2), we obtain

$$\eta' = -\mathrm{Re}\,\varphi^2 - (2\eta\xi + 2\pi\,\mathrm{Re})/(1+\xi^2),$$
$$\varphi' = \eta, \ \pi' = (2\eta - 2\pi\,\mathrm{Re}\,\xi)/\mathrm{Re}(1+\xi^2). \quad (4)$$

The system of equations (4) can be applied successfully to calculate the flow in a two-dimensional diffuser. We consider a diffuser with an angular opening of 2α ($\tan\alpha = a$). To the system of equations (4) we add the boundary conditions

$$-a \leqslant \xi \leqslant a, \ \varphi(-a) = 0, \ \pi(-a) = 0, \ \eta(-a) = c. \quad (5)$$

The problem (4), (5) has a unique solution. In making numerical calculations, we select the coefficient c from the condition $\varphi(a) = 0$. Typical profiles of the unknown functions are displayed in Fig. 1 for Re=100 and a = 0.5. The streamlines in this case are the lines ξ =const.

We note that for small angular openings of the diffuser the second derivatives with respect to the longitudinal coordinate are small. Actually, if we denote the width of the diffuser between inlet and outlet sections ($\Delta l = 2\tan\alpha\Delta x$) by l, and the mean mass velocity by u_m, we have

$$|\partial^2 u/\partial x^2| : |\partial^2 u/\partial y^2| \sim (8u_m/l^2)\,\mathrm{tg}^2\,\alpha : (16/3)\,u_m/l^2 \sim \mathrm{tg}^2\alpha.$$

Using Table 1, we can also construct various partially invariant solutions. For example, we seek a partially invariant solution of rank 1 of type 3 on the subgroup H_1 in the form

$$u = (1/x)\varphi(px^2), \ v = (1/x)\psi(px^2).$$

Proceeding now to a study of the linearized solutions of the system of equations (1), we consider a mixed problem for the flow in the duct $x \in [0, \infty]$, $y \in [-1, 1]$. We seek a solution in the form

$$u = (1-y^2) + u_1, \ v = v_1, \ p = -\frac{2x}{\mathrm{Re}} + p_1, \ \int_{-1}^{1} u_1 dy = 0, \quad (6)$$

which corresponds to the Reynolds number Re, determined from the maximum velocity in Poiseuille flow and the duct half-width. We show that $\lim_{x\to\infty} u_1 = \lim_{x\to\infty} v_1 = \lim_{x\to\infty} p_1 = 0$; i.e., we show that the solution of the linearized generalized equations of Prandtl for a two-dimensional tube tends toward Poiseuille flow.

TABLE 1

N	R	t	δ	ρ	μ	Form of the solution
1	1	4	0	1	3	$I_1, I_2, I_3, (I_4)$
2	2	3	0	0	3	$I_i=C_i, i=1,\ldots, 3$
3	2	3	1	1	2	$I_1, I_2, (I_3)$
4	3	2	1	0	2	$I_i=C_i, i=1,2$
5	3	2	2	1	1	$I_1 (I_2)$
6	4	1	2	0	1	$I_1=C$

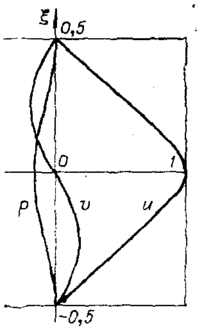

Fig. 1

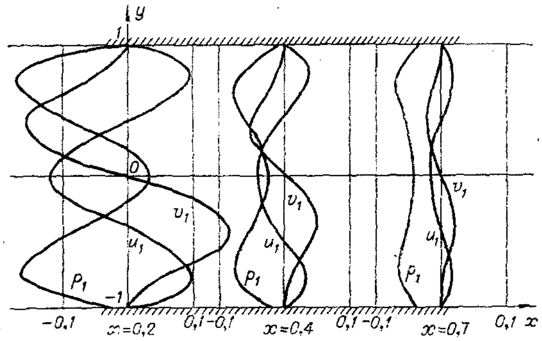

Fig. 2

Substituting the expressions (6) into the system of equations (1), assuming u_1, v_1, $p_1 \ll 1$, and neglecting the squared terms, we obtain the linear system of equations

$$(1-y^2)\partial u_1/\partial y - 2yv_1 = -\partial p_1/\partial x + (1/\mathrm{Re})\partial^2 u_1/\partial y^2;$$
$$(1-y^2)\partial v_1/\partial x = -\partial p_1/\partial y + (1/\mathrm{Re})\partial^2 v_1/\partial y^2; \quad \partial u_1/\partial x + \partial v_1/\partial y = 0. \tag{7}$$

We take the following initial and boundary conditions for the solutions of the system (7):

$$u_1(x, \pm 1) = v_1(x, \pm 1) = 0; \; u_1(0, y) = f(y); \; f(y) \in C_0[-1, 1]. \tag{8}$$

We seek a solution of the problem (7), (8) in the form

$$u_1 = \sum_{n=1}^{\infty} A_n e^{-\lambda_n x} u_1^n(y), \; v_1 = \sum_{n=1}^{\infty} B_n e^{-\lambda_n x} v_1^n(y),$$
$$p_1 = \sum_{n=1}^{\infty} e^{-\lambda_n x} p_1^n(y). \tag{9}$$

Let the set of characteristic functions $\{v_1^n\}$ and characteristic values $\{\lambda_n\}$ of the boundary-value problem, which we formulate below, be known. Then, putting $u_1^n = -(v_1^n)'$ (in accordance with the continuity equation), we have

$$B_n = -\lambda_n A_n, \; p_1^n = B_n \int_{-1}^{y} \left[-\lambda_n (1-y^2) v_1^n + \frac{1}{\mathrm{Re}} (v_1^n)''\right] dy.$$

The coefficients A_n are obtained from the infinite system of linear algebraic equations

$$\sum_{n=1}^{\infty} A_n \int_{-1}^{1} u_1^n u_1^k dy = \int_{-1}^{1} f(y) u_1^k dy, \; k = 1, \ldots, \infty.$$

The existence of a unique solution of this problem is guaranteed by the completeness of the set of characteristic functions of the corresponding boundary-value problem.

From the relations (7), (9) we obtain the equations

$$A_n u_1^n (1-y^2)\lambda_n^2 + 2y\lambda_n v_1^n B_1 + \lambda_n^2 p_1^n = -\frac{1}{\mathrm{Re}}(v_1^n)'' B_n;$$

$$-\lambda_n (1-y^2) B_n v_1^n = -(p_1^n)' + \frac{1}{\mathrm{Re}}(v_1^n)'' B_n$$

and thus arrive at the following boundary-value problem:

$$\frac{1}{\mathrm{Re}}(v_1^n)^{\mathrm{IV}} + (v_1^n)''\left[\lambda_n(1-y^2) + \frac{\lambda_n^2}{\mathrm{Re}}\right] + v_1^n\left[\lambda_n^3(1-y^2) + 2\lambda_n\right] = 0,$$
$$v_1^n(\pm 1) = (v_1^n(\pm 1))' = 0,\ n = 1, \ldots, \infty. \tag{10}$$

We now make a study of the first characteristic values of the problem (10) for large Reynolds numbers Re. Putting $\mu_n = \lambda_n$ Re and neglecting the terms μ^2/Re^2 and μ^3/Re^2, we obtain, in place of Eqs. (10),

$$(v_1^n)^{\mathrm{IV}} + \mu_n\left[(v_1^n)''(1-y^2) + 2v_1^n\right] = 0. \tag{11}$$

A similar differential equation was obtained in [6]. The boundary-value problem for Eq. (11) can be studied by the Schwarz method [Eq. (11) has the form $M|v| = \mu N|v|$]. This yields an infinite set of characteristic functions and (correspondingly) an infinite spectrum of characteristic values. The sequence is increasing; an approximate calculation yields the value $\mu_1 = 13.9$. Thus, we have Poiseuille flow as the limiting solution of the problem (7), (8). As can be seen from Eqs. (9), there is an e-fold drop in deviations from the parabolic profile at a distance $x = \mathrm{Re}/13.9$. For Re = 25, we see that $x \simeq 0.9$ of the tube diameter. This is in good agreement with the numerically calculated result given in [7]. The value given in [6] is roughly twice as large.

Finally, we consider a Cauchy problem (or a mixed boundary-value problem) for the system of generalized equations of Prandtl in the form (2). Because of the absence in the first and third equations of the corresponding second derivatives with respect to the coordinate y, the given system of equations is not parabolic in the Petrovskii sense (we are not speaking of the presence of the nonlinear terms). Therefore, in devising a numerical scheme to calculate the flow from the initial to the final section, it is necessary for stability of the solution to confirm the contractive nature of the operator of the problem. We apply the method of splitting (see [7, 8]), modified to accomodate the case in which the first part of the splitting operator is parabolic. The second part lends itself, in fact, to an analytical study. We write out the sufficient conditions for the correctness of the analytic Cauchy problem corresponding to the second operator, these conditions being necessary for a realization of a numerical solution. In addition, we apply to the system (2) the well-known method of parabolic regularization. We show the sufficiency of the stability analysis given below by means of a numerical experiment.

Thus, the transition from the previous layer to the following layer along the evolutionary coordinate is carried out in the system of equations (2) in two stages. In the first half-step we solve the evolutionary problem

$$\frac{\partial}{\partial x}\begin{pmatrix} p \\ v \\ u \end{pmatrix} = \begin{pmatrix} \frac{\beta}{\mathrm{Re}}\frac{\partial^2}{\partial y^2} & (u-\alpha)\frac{\partial}{\partial y} & -v\frac{\partial}{\partial y} \\ -\left(\frac{1}{u}+b\right)\frac{\partial}{\partial y} & \frac{1}{u\mathrm{Re}}\frac{\partial^2}{\partial y^2} - \frac{v}{u}\frac{\partial}{\partial y} & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} p \\ v \\ u \end{pmatrix} \equiv A_1\begin{pmatrix} p \\ v \\ u \end{pmatrix} \tag{12}$$

with corresponding boundary conditions, and in the second half-step we have the operator

$$\frac{\partial}{\partial x}\begin{pmatrix} p \\ v \\ u \end{pmatrix} = \begin{Vmatrix} \frac{\beta}{\mathrm{Re}}\frac{\partial^2}{\partial y^2} & \alpha\frac{\partial}{\partial y} & \frac{1}{\mathrm{Re}}\frac{\partial^2}{\partial y^2} \\ b\frac{\partial}{\partial y} & 0 & 0 \\ 0 & -\frac{\partial}{\partial y} & 0 \end{Vmatrix}\begin{pmatrix} p \\ v \\ u \end{pmatrix} \equiv A_2\begin{pmatrix} p \\ v \\ u \end{pmatrix}.$$

Here α, β, b = const. Obviously, as $\beta \to 0$, we have $A_1 + A_2 = A$, where A is the complete evolutionary operator of the system of equations (2).

It is a known fact (see [8]) that an error at a single step due to splitting is proportional to the quantity $\|A_1A_2\|\, h_x^2$, where h_x is the step along the evolutionary coordinate. Thus, when both operators are bounded, we can attain a very high degree of accuracy. It is clear that the system (12) represents a parabolic system of equations in the two unknown functions p and v. Such systems were studied in [9] for sufficiently "good" multipliers of the derivatives on the right-hand side. In numerical calculations it is, in point of fact, the linear problem that is solved, wherein the multipliers of the derivatives are taken from the previous layer. This results in an error at a given step of about h_x^2.

In making the calculations it is useful to confirm that the operator A_1 is a contraction operator. We now examine the operator A_2. Making the change of variables (6) in Eqs. (13), we obtain

$$\begin{gathered}\partial p_1/\partial x = (1/\mathrm{Re})\partial^2 u_1/\partial y^2 + (\beta/\mathrm{Re})\partial^2 p_1/\partial y^2 + \alpha \partial v_1/\partial y;\\ \partial v_1/\partial x = b\partial p_1/\partial y;\ \partial u_1/\partial x = -\partial v_1/\partial y.\end{gathered} \tag{14}$$

To study the behavior of the solutions of the system (14), we can now apply the Fourier method. We have

$$\begin{gathered}\lambda p_1^1 = \frac{1}{\mathrm{Re}} u_1^{1''} + \frac{\beta}{\mathrm{Re}} p_1^{1''} + \alpha v_1^{1'};\ \lambda v_1^1 = b p_1^{1'};\\ \lambda u_1^1 = -v_1^{1'}\ (y \to p_1^1, v_1^1, u_1^1).\end{gathered}$$

Eliminating the functions u_1^1 and p_1^1, we find that v_1^1 satisfies the ordinary differential equation

$$v_1^{1\,\mathrm{IV}} - \left(\frac{\lambda^2\beta/\mathrm{Re} + \lambda\alpha b}{b}\mathrm{Re}\right)v_1^{1''} + \frac{\lambda^3}{b}\mathrm{Re}\, v_1^1 = 0.$$

Including the boundary conditions for this equation, for example, in the form (10), we find that the boundary-value problem is solvable if the roots of the characteristic equation

$$z^4 - \left(\frac{\lambda^2\beta/\mathrm{Re} + \lambda\alpha b}{b}\mathrm{Re}\right)z^2 + \frac{\lambda^3}{b}\mathrm{Re} = 0$$

are of the form $z_{1,2} = \pm i\xi$, $z_{3,4} = \pm\eta$, where ξ is a real number, η is an arbitrary complex number connected with ξ by means of the condition for the vanishing of the characteristic determinant. Putting $\xi^2 = C$, $-\eta^2 = c$, we find from Vieta's theorem a relationship among the characteristic numbers λ and z_k, $k = 1, \ldots, 4$:

$$-\lambda^2 s - \lambda t = C + c;\ q\lambda^3 = Cc, \tag{15}$$

where $q = \mathrm{Re}/b$; $s = \beta/b$; $t = \alpha\ \mathrm{Re}$. From the relations (15) we obtain an algebraic equation in the number λ:

$$\lambda^3 + \lambda^2 sC/q + \lambda tC/q + C^2/q = 0. \tag{16}$$

Thus, the question as to the correctness of the Cauchy problem for the operator A_2 is reduced to a study of the roots of Eq. (16). The condition Real $\lambda_i < 0$ is satisfied only when

$$sC/q > 0,\ tC/q > 0,\ C^2/q > 0,\ (sC/q)(tC/q) > C^2/q.$$

This gives the condition for the correctness of the evolutionary problem for the operator A_2 in the form

$$\alpha\beta > 1. \tag{17}$$

Through an appropriate choice of the splitting the condition (17) becomes independent of the spectrum of the characteristic numbers of the boundary-value problem for

$$v_1^1 : \{C_n\},\ n = 1, \ldots, \infty.$$

The condition (17), however, imposes an essential restriction on the step size h_x. Indeed, since $\|A_1A_2\| \sim \alpha^2$, the error made in the calculations in going from a given layer to the following layer may be estimated by means of the relationship

$$\Delta \sim \|A_1A_2\| h_x^2 \sim \alpha h_x^2 \sim \left(\frac{hx}{\beta}\right)^2.$$

Thus, to preserve accuracy in the calculations as $\beta \to 0$, we must refine the step size h_x.

We proceed now to a numerical solution of the problem concerning fluid flow in a two-dimensional duct. This problem was solved in a simplified form in [10]. To the system of equations (2) we adjoin the following conditions:

$$\begin{gathered}u(x, \pm 1) = v(x, \pm 1) = 0,\ u(0, y) = u_0(y),\ v(0, y) = v_0(y),\\ p(0, y) = p_0(y);\ x \in [0, \infty],\ y \in [-1, 1];\ u_0, v_0, p_0 \in C_0[-1, 1].\end{gathered}$$

Applying the decomposition (12), (13), we consider each problem separately. The difference scheme corresponding to the first operator (12) may be written in the form

$$p_j^{i+1} - p_j^i = \frac{\tau\beta}{\mathrm{Re}\,h^2}\left(p_{j-1}^{i+1} - 2p_j^{i+1} + p_{j+1}^{i+1} + \tau\left(2u_j^i - u_j^{i-1} - \right.\right.$$

$$\left. - \alpha\right)\frac{v_{j+1}^{i+1} - v_{j-1}^{i+1}}{2h} - \tau\left(2u_{j+1}^i - 2u_{j-1}^{i-1} + u_{j-1}^{i-1} - u_{j+1}^{i-1}\right)\frac{v_j^{i+1}}{2h};$$

$$v_j^{i+1} - v_j^i = \frac{\tau}{\left(2u_j^i - u_j^{i-1}\right)\mathrm{Re}\,h^2}\left(v_{j-1}^{i+1} - 2v_j^{i+1} + v_{j+1}^{i+1}\right) -$$

$$- \tau\frac{2v_j^i - v_j^{i-1}}{2u_i^i - u_j^{i-1}}\,\frac{v_{j+1}^{i+1} - v_{j-1}^{i+1}}{2h} - \frac{\tau}{2h}\left(\frac{1}{2u_j^i - u_j^{i-1}} + b\right)\left(p_{j+1}^{i+1} - p_{j-1}^{i+1}\right),$$

$$i = 0, \ldots, \infty,\ j = 0, \ldots, 2N,\ h = 1/N,\ \tau = h_x;$$

$$v_0^i = v_{2N}^i = 0,\ p_0^{i+1} = p_0^i,\ p_{2N}^{i+1} = p_{2N}^i.$$

We solve the difference boundary-value problem by the method of drive:

$$p_j = A_{j+1}p_{j+1} + B_{j+1}v_{j+1} + C_{j+1};$$

$$v_j = D_{j+1}p_{j+1} + E_{j+1}v_{j+1} + F_{j+1},$$

which is usually employed for parabolic systems with no essential difficulties. The operator A_1 is a parabolic nonlinear operator. In all the numerical calculations its norm was found to be less than one. The numerical computation for the second part of the problem requires a special approach. We write down the difference scheme for the operator (13):

$$p_j^{i+1} - p_j^i = \frac{\tau\beta}{\mathrm{Re}\,h^2}\left(p_{j-1}^{i+1} - 2p_j^{i+1} + p_{j+1}^{i+1}\right) + \frac{\tau}{\mathrm{Re}\,h^2}\left(u_{j-1}^{i+1} - 2u_j^{i+1} - u_{j+1}^{i+1} + \frac{\alpha}{2h}\left(v_{j+1}^{i+1} - v_{j-1}^{i+1}\right)\right);$$

$$u_j^{i+1} - u_j^i + u_{j+1}^{i+1} - u_{j+1}^i = \frac{2\tau}{h}\left(v_j^{i+1} - v_{j+1}^{i+1}\right); \tag{18}$$

$$v_j^{i+1} - v_j^i + v_{j+1}^{i+1} - v_{j+1}^i = \frac{2\tau b}{h}\left(p_{j+1}^{i+1} - p_j^{i+1}\right);$$

$$u_0^i = u_{2N}^i = v_0^i = v_{2N}^i = 0.$$

As is evident from the relations (18), the last two equations are written for the fictitious intermediate point $j+1/2$. The absence of boundary conditions for p requires a corresponding increase in the equations for the numerical scheme. For this reason, we realize the predictor in the form

$$u_j = A_{j+1}u_{j+1} + B_{j+1}v_{j+1} + C_{j+1};$$

$$v_j = D_{j+1}u_{j+1} + E_{j+1}v_{j+1} + F_{j+1};$$

$$p_j = G_{j+1}u_{j+1} + H_{j+1}v_{j+1} + K_{j+1}.$$

For the start of the predictor cycle, we need to write, using the last two equations of the system (18) twice, the system of equations in the neighborhood of the boundary in the form

$$lp_0 + mp_1 + lp_2 + n(u_2 - 2u_1) + tv_2 = -p_1^0,$$

$$u_1 + \mu v_1 = \xi_0,\ u_2 + u_1 + \mu(v_2 - v_1) = \xi_1,\ v_1 + \nu(p_1 - p_0) = \eta_0,$$

$$v_2 + v_1 + \nu(p_2 - p_1) = \eta_1,$$

$$l, m, n, t, p_1^0, \mu, \xi_0, \xi_1, \nu, \eta_0, \eta_1 = \text{const};$$

from these equations we can express p_0, p_1, p_2, v_1, u_1 in terms of v_2 and u_2, and this stipulates the start of the predictor process. In the course of the calculation, we eliminate the quantity p_{j+1} in the first equation of (18) with the help of the third equation. In the rest of the calculations the predictor process is the standard one. The possibility of constructing a noncontradictory scheme, one which at first glance appears to be indeterminate, is guaranteed by the fact that the system of equations (13) is of the fourth order in the variable y.

In making the calculations, we printed out, after a specified number of steps, the grid norm in L_2 of the unknown functions after each of the operators A_i. Thus, we verified the contractive nature of both operators numerically. To improve the stability of the solution in the case of large Reynolds numbers, we carried out a repeated iteration at the second half-step.

Up to Reynolds numbers of order 1000 the whole computational scheme was found to be stable in the large. The solution of the problem has a Poiseuille profile as its limiting profile. Typical profiles are shown, by section, in Fig. 2 ($p_1 = p + 2x/\mathrm{Re}$; $u_1 = u - (1 - y^2)$; $v_1 = v$; $\mathrm{Re} = 10$). It should be noted that our method permits the introduction, without any notable difficulties, of semiempirical turbulent stresses into the calculations.

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LITERATURE CITED

1. V. M. Solonenko and A. K. Yanko, "Approximate equations for the flow of a viscous fluid applicable to an axially symmetric flow," Izv. Akad. Nauk BelorusSSR, No. 2, (1975).
2. G. R. Inger, "Strong blowing across a Couette–Poiseuille shear flow," Phys. Fluids, 12, No. 9, 1741-1746 (1969).
3. Stefan Loer, "Examination of the stability of disturbed boundary layer flow by a numerical method," in: IUTAM Symposium Proceedings, Monterey, California, August (1968).
4. L. V. Ovsyannikov, Group Properties of Differential Equations [in Russian], Izd. Sibirsk Otd., Akad. Nauk SSSR, Novosibirsk (1962).
5. N. Kh. Ibragimov, Group Properties of Some Differential Equations [in Russian], Nauka, Novosibirsk (1967).
6. H. Schlichting, Boundary Layer Theory, 6th ed., McGraw-Hill, New York (1968).
7. T. D. Taylor and É. Ndefo, "Calculation of the flow of a viscous fluid in a duct with the aid of the method of splitting," in: Proceedings of the Second International Conference on Numerical Methods in Fluid Dynamics, Springer-Verlag, Berlin (1971).
8. N. N. Yanenko, Method of Fractional Steps for the Solution of Multidimensional Problems of Mathematical Physics [in Russian], Nauka, Novosibirsk (1967).
9. S. D. Éidel'man, Parabolic Systems [in Russian], Nauka, Moscow (1964).
10. A. K. Yanko, "Regular solutions of the equations of motion of a viscous fluid and calculation of the flow in a two-dimensional duct," Izv. Akad. Nauk SSSR, Sibirsk. Otd., Ser. Tekh. Nauk, No. 8 (1974).

COMPUTATION OF UNSTEADY FLOW PAST A CYLINDER INSTANTANEOUSLY SET IN MOTION

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§1. The first results on the solution of unsteady flow past a body of finite dimensions instantaneously set in motion were obtained within the framework of the boundary-layer theory.

For the initial flow stage the first two terms of the power series expansion of the solution in the powers of t (t is time) were obtained by Blasius in [1], the obtained solution being valid as $\mathrm{Re} \to \infty$.

The solution found by Blasius was improved in [2]. Subsequently, an attempt was made to extend the Blasius solution to the case of low Reynolds numbers [3, 4].

The use of numerical methods to solve nonstationary Navier–Stokes equations [5-10] turns out to be a more promising approach to the problem under investigation. In [10] a survey of the literature on this subject is given. In the case of suddenly arising motion of a cylinder one of the difficulties lies in the formulation of the initial conditions.

It follows from the theory of the boundary layer [11] that the vorticity of the fluid flow is infinitely large at the initial time instant and is then concentrated in an infinitely thin region around the cylinder surface. Therefore, a straightforward application of finite-difference approximations to the original equations does not produce a correct pattern of the initial flow past the cylinder [7]. Moreover, it was shown in [12] that to obtain in this case a satisfactory approximate solution very small steps in time must be taken.

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